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Classifying singularities up to analytic extensions of scalars is smooth

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ABSTRACT

The singularity space consists of all germs (X, x), with X a Noetherian scheme and x a point, where we identify two such germs if they become the same after an analytic extension of scalars. This is a complete, separable space for the metric given by the order to which jets (=infinitesimal neighborhoods) agree after base change. In the terminology of descriptive set-theory, the classification of singularities up to analytic extensions of scalars is a smooth problem. Over $\mathbb{C}$, the following two classification problems up to isomorphism are then also smooth: (i) analytic germs; and (ii) polarized schemes.

1. Introduction

Algebraic geometers usually approach a classification problem as a moduli question, that is to say, as a quotient of a Hilbert scheme parameterizing certain geometric objects. For this to be finite dimensional, objects can only depend on finitely many parameters. One way of controlling this is by restricting certain invariants, giving rise therefore to a sequence of moduli indexed by some natural numbers or other, concrete invariants. However, how to view this sequence as an algebraic-geometric object? We approach the issue from a different angle: instead of a geometric structure, we will put a metric structure on the objects to be classified, without restriction on their defining parameters. In our case, the geometric objects are germs of points on Noetherian schemes up to formally etale extensions. In this metric, a germ is approximated by its jets, where, the n-th *jet* $J^n\mathcal{O}_{X,x}$ is the Artinian — whence "finitary" — closed subscheme defined by the n-th power $\mathfrak{m}_x^n$ of the maximal ideal $\mathfrak{m}_x$ of the germ. The main result of this paper is that this metric is complete, where limits are given by a variant of the ultraproduct construction, to wit, the *cataproduct*. In the parlance of descriptive set-theory, we have shown that the classification of germs up to formally etale extensions is smooth.

Roughly speaking, a classification problem consists of a class of objects together with an equivalence relation telling us which objects to identify; a solution to this problem is then an 'effective' or 'concrete' description of the quotient, preferably by a system of complete invariants. What constitutes a reasonably concrete or effective solution to a classification problem, however, might depend on one's purposes or even one's taste. Descriptive set-theory proposes smoothness to be the decisive indication that a classification is explicit and/or concrete (see for instance [10,9] for a discussion). More precisely, recall that a *Polish* space is an uncountable complete metric space containing a countable dense subset. Considering a Polish space to be

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concrete is justified by the fact that its underlying Borel structure is in essence equal to the standard Borel space $\mathbb{R}$ (with $\mathbb{Q}$ as its countable dense subset). With this in mind, an equivalence relation on a Polish space, and by extension, the classification problem it encodes, is called *smooth* if there is a Borel map to a Polish space which factors through the quotient. A more suggestive, albeit slightly less precise formulation is that a classification problem is smooth if, up to a Borel (read: concrete) isomorphism, equivalence classes are completely classified by real numbers.

We now know that beyond smoothness, there exists a plethora of non-smooth classification degrees, the so-called *Borel degrees*, and much work has been done recently to map many classification problems from various mathematical fields into this still poorly understood world (e.g., [5,12,26,27,25]). As an algebraic geometer, it has always bothered me that the status of most classification problems in algebraic geometry, like classifying varieties over a fixed algebraically closed field up to isomorphism or up to bi-rational equivalence, is completely unknown. Of course, this is in no way preventing my colleagues to seriously, and often successfully, work on these classification problems.

Our main application of the jet metric is a partial answer to this general problem: *similarity* — a weaker equivalence relation where instead of isomorphisms we allow arbitrary formally etale extensions, called *analytic extensions of scalars* in this paper — is a local classification problem which falls at the right side of the dividing line: one can 'concretely', that is to say, smoothly, classify germs of points on arbitrary Noetherian schemes up to similarity. Using this general result, we do deduce smoothness results for certain abstract isomorphism problems (that is to say, defined over $\mathbb{Z}$). For *analytic germs* (= formal completions of germs in the sense of [7], Section II.9), we have:

Theorem 1.1. *Over an algebraically closed field of size the continuum, the classification of analytic germs up to (abstract) isomorphism is smooth.*

This also enables us to obtain a smooth classification problem of a more global nature, namely for projective schemes together with a choice of a very ample line bundle, the so-called *polarized schemes*.

Theorem 1.2. *The classification, up to (abstract) isomorphism, of polarized schemes over an algebraically closed field of size the continuum, is smooth.*

A word needs to be said about the choice of equivalence relation, namely, why only up to similarity. As we may associate to a germ its local ring, the problem reduces to the classification of Noetherian local rings. If we were to classify these only up to isomorphism, then as part of this problem, we would already have to classify all fields, and even for countable fields [5] or fields of finite transcendence degree [27] these are non-smooth problems. Hence to circumvent this arithmetical obstruction, we can either fix the residue field — the route taken for the two isomorphism problems stated above — or, otherwise, allow for 'extensions of scalars', resulting in the identification of any two fields of the same characteristic. Even after taking the latter modification, the local classification problem is probably still not smooth. For many problems in algebraic geometry, one prefers to work in the etale topos instead of the (classical) Zariski topos. This translates into the observation that two local rings can be considered identical if they have a common etale extension, or more generally, if they have the same completion. In conclusion, we say that two Noetherian local rings are *similar* if they can be made isomorphic by an *analytic extension of scalars*, that is to say, by the process of extending scalars and taking completion. To also make sense of this in mixed characteristic, we subsume these types of extensions under the larger class consisting of all formally etale (=unramified and faithfully flat) extensions. We show that *similar* points (meaning that their corresponding local rings are similar) have the same type of singularity (see Theorem 4.1). As a spinoff of this investigation, we obtain a flatness criterion generalizing a result of Kollár:

Theorem 3.14. *Let $R \to S$ be a local homomorphism between Noetherian local rings and suppose R is an excellent normal domain with perfect residue field. If $\dim(R) = \dim(S)$ and $R \to S$ is unramified, then $R \to S$ is faithfully flat.*

Our method to obtain smoothness is highly abstract: a priori, we can only associate, in a Borel fashion, to any equivalence class a unique real number. Due to the inherent self-referential nature of algebraic geometry — e.g., classification via moduli spaces — one can perhaps do better than just Borel, and so, tentatively, any such assignment which is moreover continuous is called a *slope* (note that a Borel map is only continuous outside a meager set). For any of the classification problems discussed here, does there exist a countable filtration, preferably given by natural invariants, such that each piece admits a slope? The main contribution of the present theory to finding explicit complete invariants now is the fact that a slope is completely determined by its values on the class of Artinian local rings, so that it suffices to restrict one's quest to this latter class. If we omit the requirement that the assignment is injective, we get the notion of a pre-slope. We show in the last section how many of the classical invariants, like dimension and Euler characteristic, can be derived as pre-slopes on certain filtrations.

2. Limits and ultraproducts

Let (Σ, d) be a *semi-metric space*. In this paper, we understand this to mean that the semi-metric, which takes values in the reals, is *non-archimedean*, that is to say, $d(x, y) \leq \max\{d(x, z), d(y, z)\}$ for all $x, y, z \in \Sigma$, and *bounded*, that is to say, after possibly normalizing the metric, $d(x, y) \leq 1$ for all $x, y \in \Sigma$. We call d a *metric*, if $d(x, y) = 0$ if and only if $x = y$. To include the jet metric in our treatment, we allow for Σ to be merely a class. We say that two elements $r, s \in \Sigma$ are *d-equivalent*, written $r \sim_d s$, if $d(r, s) = 0$. The quotient space $\Sigma/\sim_d$ has an induced semi-metric which is in fact a metric; we therefore call this quotient the *metrization* of (Σ, d).

Let (Σ_w, d_w) be semi-metric spaces, for $w \in \mathbb{N}$. We will identify the elements of the product $\Pi := \prod_w \Sigma_w$ with the sequences $\mathbf{r}: \mathbb{N} \to \Pi$ such that $\mathbf{r}(w) \in \Sigma_w$ for each w. The *product semi-metric* on Π is given by letting the distance $d(\mathbf{r}, \mathbf{s})$ between two sequences $\mathbf{r}$ and $\mathbf{s}$ be the lim-inf of the distances $d_w(\mathbf{r}(w), \mathbf{s}(w))(\leq 1)$ of their respective components. Below, we will introduce weaker semi-metrics on Π, induced by ultrafilters.

2.1. Cauchy sequences

Let $\mathbf{r}$ be a sequence in Σ (meaning that all $\mathbf{r}(w) \in \Sigma$) and let $\mathbf{r}^+$ be its *twist*, given as the sequence whose w-th element is $\mathbf{r}(w+1)$. We call $\mathbf{r}$ a *Cauchy sequence* if $\mathbf{r} \sim \mathbf{r}^+$ (with respect to the product semi-metric). One verifies that $\mathbf{r}$ is a Cauchy sequence, if for each $\varepsilon > 0$, there exists an N such that $d(\mathbf{r}(w), \mathbf{r}(v)) < \varepsilon$ for all $v, w > N$, and that two Cauchy sequences $\mathbf{r}$ and $\mathbf{s}$ are equivalent if for each $\varepsilon > 0$, there exists an N such that $d(\mathbf{r}(w), \mathbf{s}(w)) < \varepsilon$ for all $w > N$. Let $\mathrm{Cau}(\Sigma, d)$, or simply, $\mathrm{Cau}(\Sigma)$, denote the set of all Cauchy sequences in Σ with the induced product semi-metric. There is a natural isometry $\Sigma \to \mathrm{Cau}(\Sigma)$ sending x to the constant sequence $\mathbf{x}$ given as $\mathbf{x}(w) := x$; we will identify the element x with its constant sequence in $\mathrm{Cau}(\Sigma)$.

A *limit* of a sequence $\mathbf{r}$ is an element $x \in \Sigma$ such that $\mathbf{r} \sim x$. It is easy to see that if $\mathbf{r}$ has a limit, then it must be Cauchy. We call (Σ, d) *complete* if every Cauchy sequence has a unique limit. This implies in particular that d is a metric. We define the *completion* of (Σ, d) as the metrization $\widehat{\Sigma} := \mathrm{Cau}(\Sigma)/\sim$ of the semi-metric space $\mathrm{Cau}(\Sigma)$; it is a complete metric space containing Σ as a dense subspace.

2.2. Adic metric

A local ring $(R, \mathfrak{m})$ comes with a canonical semi-metric, its $\mathfrak{m}$*-adic semi-metric* defined as follows: the *order* of an element $x \in R$ is the supremum of all n for which $x \in \mathfrak{m}^n$; the distance $d_R(x, y)$ between two elements is then equal to 2^{-n} where n is the order of $x - y$ (we allow the case $n = \infty$, with the convention that $2^{-\infty} = 0$). The subset of all elements which are d_R-equivalent to zero forms an ideal, equal to the intersection of all the powers $\mathfrak{m}^n$; it is called the *ideal of infinitesimals* of R and is denoted $\mathrm{Inf}(R)$. By Krull's intersection theorem, if R is Noetherian, then $\mathrm{Inf}(R) = 0$. The completion of R in the $\mathfrak{m}$-adic semi-metric will be denoted $\widehat{R}$. If R has finite embedding dimension, then $\widehat{R}$ is a complete Noetherian local ring by [22, Theorem 2.2].

Below, we will define a semi-metric on the class of all Noetherian local rings, not to be confused with the adic metric on a single Noetherian local ring. To calculate limits in the former semi-metric, we need a notion from model-theory: the ultraproduct construction (some references for ultraproducts are [4,11,16,23], or the brief reviews in [18, Section 2] and [21]).

2.3. Ultraproducts and cataproducts

Let $(R_w, \mathfrak{m}_w)$, for $w \in \mathbb{N}$, be a sequence of Noetherian local rings. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, which we always assume to be non-principal. The *ultraproduct* of the R_w with respect to $\mathcal{U}$, denoted $R_\natural$, is obtained from the product $\Pi := \prod_w R_w$ by modding out the ideal of all sequences almost all of whose entries are zero (it is customary to use the expression "almost all" to mean "all indices belonging to a member of the ultrafilter"). The particular choice[1] of ultrafilter $\mathcal{U}$ does not matter for our purposes, and hence we do not include it in our notation. Although not useful for proving results, let me recall an alternative construction from [23, Theorem 2.5.4]: there exists a minimal prime ideal $\mathfrak{g}$ of the Cartesian power $\mathbb{Z}^{\mathbb{N}}$, containing the direct sum ideal $\mathbb{Z}^{(\mathbb{N})}$, such that $R_\natural = \Pi/\mathfrak{g}\Pi$, where we view the Cartesian product Π as an algebra over $\mathbb{Z}^{\mathbb{N}}$ in the natural way; and conversely, any such prime ideal determines in this way an ultraproduct of the R_w.

The ultraproduct $R_\natural$ is again a local ring, with maximal ideal $\mathfrak{m}_\natural$ given as the ultraproduct of the $\mathfrak{m}_w$. In general, however, $R_\natural$ will no longer be Noetherian. If almost all R_w have embedding dimension at most n, then so does $R_\natural$. A key role will played by the homomorphic image of $R_\natural$ modulo its ideal of infinitesimals $\mathrm{Inf}(R_\natural)$, which we call the *cataproduct* of the R_w and which we denote by $R_\sharp$. A more direct way for defining the cataproduct, although less useful in proofs, is as follows: on the product Π, the ultrafilter $\mathcal{U}$ induces a semi-metric $d_{\mathcal{U}}$ by the condition that $d_{\mathcal{U}}(\mathbf{r}, \mathbf{s}) \leq \varepsilon$ for some ε if and only if $d_{R_w}(\mathbf{r}(w), \mathbf{s}(w)) \leq \varepsilon$ for almost all w. The cataproduct is then the metrization of $(\Pi, d_{\mathcal{U}})$, that is to say, the residue ring of the product modulo the ideal of all sequences which are $d_{\mathcal{U}}$-equivalent to zero.

If almost all R_w have embedding dimension at most n, then so does the cataproduct $R_\sharp$. Moreover, by the saturatedness property of ultraproducts, the cataproduct is a complete local ring, whence Noetherian by [14, Theorem 29.4] (for more details see [22, Lemma 5.6] or [23, Theorem 12.1.4]). The same argument also shows that the R_w and their completions $\widehat{R_w}$ have the same cataproduct.

We will only consider cataproducts of Noetherian local rings of bounded embedding dimension, so that we tacitly may assume that they are complete and Noetherian. In case all R_w are equal to a fixed Noetherian local ring R, then their

[1] There is really no reason to restrict only to ultraproducts on a countable index set, although it is the only type we will use in this paper. However, for the cataproduct (see below) to be Noetherian and complete, we do have to impose that the ultrafilter be countably incomplete, which automatically holds on countable index sets and always exists on arbitrary index sets.

ultraproduct $R_\natural$ and cataproduct $R_\sharp$ are called, respectively, the *ultrapower* and *catapower* of R. By Łoś' Theorem, ultrapowers commute with base change, that is to say $(R/I)_\natural \cong R_\natural/IR_\natural$; the same is true for catapowers by [22, Corollary 5.7] or [23, Corollary 8.1.13]:

Lemma 2.4. *If R is a Noetherian local ring and I an ideal in R, then $(R/I)_\sharp = R_\sharp/IR_\sharp$.*

3. Scalar extensions

Cohen's structure theorems for complete Noetherian local rings will play an essential role in this paper, so we quickly review the relevant properties; a good reference for all this is [14, Section 29]. For each field κ of prime characteristic p, there exists a unique complete discrete valuation ring V of characteristic zero whose residue field is κ and whose maximal ideal is pV; we call V the *complete p-ring* over κ. Let R be a Noetherian local ring with residue field κ. We say that R has *equal characteristic* if R and κ have the same characteristic; in the remaining case, we say that R has *mixed characteristic*. Assume R is moreover complete and let X be a finite tuple of indeterminates. Cohen's structure theorems now claim, among other things, the following:

- if R has equal characteristic, then it is a homomorphic image of $\kappa[[X]]$;
- if R has mixed characteristic, then it is a homomorphic image of $V[[X]]$, where V is the complete p-ring over κ.

3.1. Scalar extensions

Let $(R, \mathfrak{m})$ be a Noetherian local with residue field κ and let λ be a field extension of κ. With a *scalar extension of R over λ* we mean a local R-algebra $(S, \mathfrak{n})$ with residue field λ such that $R \to S$ is faithfully flat, $\mathfrak{n} = \mathfrak{m}S$ and $R \to S$ induces the embedding $\kappa \subseteq \lambda$ on the residue fields. A *scalar extension* of a local ring R is then a scalar extension of R over some field extension of its residue field. The condition that $\mathfrak{n} = \mathfrak{m}S$ is also expressed by saying that $R \to S$ has *trivial closed fiber* or that it is *unramified*. In other words, a scalar extension is the same as an unramified, faithfully flat homomorphism (also called a *formally etale* extension). By [6, 0_{III} 10.3.1], for any Noetherian local ring R and any extension l of its residue field, at least one scalar extension of R over l exists; we will reprove this in Corollary 3.5 below.

Proposition 3.2. *Consider the following commutative triangle of local homomorphisms between Noetherian local rings*

$$\begin{tikzcd} & (R, \mathfrak{m}) \arrow[dl, "f"'] \arrow[dr, "h"] & \\ (S, \mathfrak{n}) \arrow[rr, "g"'] & & (T, \mathfrak{p}) \end{tikzcd} \tag{1}$$

If any two are scalar extensions, then so is the third.

Proof. It is clear that the composition of two scalar extensions is again scalar. Assume g and h are scalar extensions. Then f is faithfully flat and $\mathfrak{m}T = \mathfrak{p} = \mathfrak{n}T$. Since g is faithfully flat, we get $\mathfrak{m}S = \mathfrak{m}T \cap S = \mathfrak{n}T \cap S = \mathfrak{n}$, showing that f is also a scalar extension. Finally, assume that f and h are scalar extensions. Let

$$\dots R^{b_2} \to R^{b_1} \to R \to R/\mathfrak{m} \to 0 \tag{2}$$

be a free resolution of $R/\mathfrak{m}$. Since S is flat over R, tensoring yields a free resolution

$$\dots S^{b_2} \to S^{b_1} \to S \to S/\mathfrak{m}S \to 0. \tag{3}$$

By assumption $S/\mathfrak{m}S$ is the residue field λ of S. Therefore, $\operatorname{Tor}^S_\bullet(T, \lambda)$ can be calculated as the homology of the complex

$$\dots T^{b_2} \to T^{b_1} \to T \to T/\mathfrak{m}T \to 0 \tag{4}$$

obtained from (3) by the base change $S \to T$. However, (4) can also be obtained by tensoring (2) over R with T. Since T is flat over R, the sequence (4) is exact, whence, in particular, $\operatorname{Tor}^S_1(T, \lambda) = 0$. By the local flatness criterion, T is flat over S. Since $\mathfrak{n} = \mathfrak{m}S$ and $\mathfrak{p} = \mathfrak{m}T$, we get $\mathfrak{p} = \mathfrak{n}T$, showing that g, too, is a scalar extension. □

Three important examples of scalar extensions are given by the following proposition.

Proposition 3.3. *Let R be a Noetherian local ring.*

1. *The natural map $R \to \widehat{R}$ is a scalar extension.*
2. *Any etale map is a scalar extension.*
3. *The natural map $R \to R_\sharp$ is a scalar extension, where $R_\sharp$ is any catapower of R.*

Proof. The first two assertions are well-known, so remains to show the last. Let $\mathfrak{m}$ be the maximal ideal of R. It is easy to show that $\mathfrak{m}R_\sharp$ is the maximal ideal of $R_\sharp$. So remains to prove that $R \to R_\sharp$ is flat. Since $R_\sharp$ is complete, and in fact equal to the catapower of $\widehat{R}$, we may assume without loss of generality that R is already complete. In particular, R is a homomorphic image of a regular local ring and if we prove the corresponding result for this regular local ring, then a base change yields the desired result by Lemma 2.4. Therefore, we may moreover assume that R is regular. Since $\mathfrak{m}R_\sharp$ is the maximal ideal of $R_\sharp$ and since $R_\sharp$ is also regular by [22, Corollary 5.15] or [23, Corollary 8.1.14], of the same dimension as R, the flatness of $R \to R_\sharp$ then follows from [14, Theorem 23.1]. $\square$

In fact, Proposition 3.3.2 has the following converse: if $R \to S$ is essentially of finite type inducing a finite separable extension on the residue fields, then $R \to S$ is a scalar extension if and only if it is etale. In this sense, scalar extensions are generalizations of etale maps (whence the alternative terminology 'formally etale' for them). This shows already that classification up to scalar extension is a reasonable and interesting problem. To gather further support for this claim, we will now explore how closely related scalar extensions are to isomorphisms. An important observation in that direction, one we will use several times below, is that a scalar extension of complete Noetherian local rings inducing an isomorphism on their residue fields is itself an isomorphism; see [14, Theorem 8.4]. Hence it is of interest to generate scalar extensions $R \to S$ with S complete. We will see that there exists a canonical choice over any field.

3.4. *Completions along a residual extension*

Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field κ, and let λ be a field extension of κ. The *completion of R along* λ is the (unique) local R-algebra $R^\wedge_\lambda$ solving the following universal problem: given an arbitrary Noetherian local R-algebra S with residue field λ, if S is complete, then there exists a unique local R-algebra homomorphism $R^\wedge_\lambda \to S$. When $\kappa = \lambda$, we recover the usual completion $R^\wedge_\kappa = \widehat{R}$ of R. Here and elsewhere, we say that there is a *unique* homomorphism with certain properties, when we actually mean that there exists a unique homomorphism *up to isomorphism*; this is consistent with our practice of identifying two local rings when they are isomorphic.

Proof of the existence of a completion along λ. We have to treat the equal and mixed characteristic cases separately. Assume first that R has equal characteristic (this case is also discussed in [8, (6.3)]). By Cohen's structure theorems, there exists an embedding $\kappa \to \widehat{R}$. Let $R^\wedge_\lambda$ be the $\mathfrak{m}(\widehat{R} \otimes_\kappa \lambda)$-adic completion of $\widehat{R} \otimes_\kappa \lambda$. To see that this is a completion along λ, let S be a Noetherian local R-algebra with residue field λ and assume that S is complete. By the universal property of ordinary completions, we get a unique homomorphism $\widehat{R} \to S$. Since S is complete, we can find an embedding $\lambda \to S$ which agrees on the subfield κ of λ with the composition $\kappa \to \widehat{R} \to S$. By the universal property of tensor products, the two maps $\widehat{R} \to S$ and $\lambda \to S$ combine to a unique local R-algebra homomorphism $\widehat{R} \otimes_\kappa \lambda \to S$, and using once more the universal property of completion, this then yields a unique R-algebra homomorphism $R^\wedge_\lambda \to S$.

In the mixed characteristic case, coefficient fields no longer exist and we now proceed as follows. Let V be the (unique) complete p-ring over κ, where p is the characteristic of κ. We first define the completion of V along λ, that is to say, $V^\wedge_\lambda$, as the unique complete p-ring over λ. That the latter satisfies the universal property of a completion along λ is proven in [14, Theorem 29.2]. To define $R^\wedge_\lambda$, let S be any Noetherian local R-algebra with residue field λ extending κ, and assume S is complete. As before, we have a unique local R-algebra homomorphism $\widehat{R} \to S$. By Cohen's structure theorems, there exists a commutative diagram of local homomorphisms

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V^\wedge_\lambda \arrow[d] \\ \widehat{R} \arrow[r] & S. \end{tikzcd} \tag{5}$$

By the universal property of tensor products, we get a unique R-algebra homomorphism $\widehat{R} \otimes_V V^\wedge_\lambda \to S$. Define $R^\wedge_\lambda$ now as the $\mathfrak{m}(\widehat{R} \otimes_V V^\wedge_\lambda)$-adic completion of $\widehat{R} \otimes_V V^\wedge_\lambda$, so that we get a unique local R-algebra homomorphism $R^\wedge_\lambda \to S$, as required. $\square$

Corollary 3.5. *For every Noetherian local ring R and every extension field* λ *of its residue field,* $R^\wedge_\lambda$*, the completion of R along* λ*, exists and is unique. For every ideal I in R, the completion of R/I along* λ *is equal to* $R^\wedge_\lambda / IR^\wedge_\lambda$*.*

Moreover, the natural map $R \to R^\wedge_\lambda$ *is a scalar extension over* λ*.*

Proof. Existence was proven above; uniqueness then follows formally from being a solution to a universal problem. To prove the second assertion, assume $R/I \to S$ is a local homomorphism with S a complete Noetherian local ring with residue field λ. The composition $R \to R/I \to S$ yields by definition a unique local R-algebra homomorphism $R^\wedge_\lambda \to S$. Since $IS = 0$, the latter homomorphism factors through $R^\wedge_\lambda / IR^\wedge_\lambda$, showing that $R^\wedge_\lambda / IR^\wedge_\lambda$ satisfies the universal property of completions along

λ. As for the last assertion, in the equal characteristic case, the base change $\widehat{R} \to \widehat{R} \otimes_\kappa \lambda$ of $\kappa \subseteq \lambda$ is faithfully flat. Since completion is exact, each map in

$$R \to \widehat{R} \to \widehat{R} \otimes_\kappa \lambda \to R_\lambda^\wedge$$

is faithfully flat, whence so is their composition. In the mixed characteristic case, $V_\lambda^\wedge$ is torsion-free whence flat over V. Hence by the same argument as in the equal characteristic case, the composite map

$$R \to \widehat{R} \to \widehat{R} \otimes_V V_\lambda^\wedge \to R_\lambda^\wedge$$

is faithfully flat. By our second assertion, $R_\lambda^\wedge/\mathfrak{m}R_\lambda^\wedge$ is the completion of $R/\mathfrak{m} \cong \kappa$ along λ. In other words, $R_\lambda^\wedge/\mathfrak{m}R_\lambda^\wedge \cong \lambda$ and hence in particular, $\mathfrak{m}R_\lambda^\wedge$ is the maximal ideal of $R_\lambda^\wedge$. This proves that $R \to R_\lambda^\wedge$ is a scalar extension. □

Proposition 3.6. *Let $R \to S$ be a scalar extension over λ. If S is complete, then $S \cong R_\lambda^\wedge$.*

Proof. By the universal property, we have a local R-algebra homomorphism $R_\lambda^\wedge \to S$. It follows from [14, Theorem 8.4] that $R_\lambda^\wedge \to S$ is surjective. Since $R \to R_\lambda^\wedge$ and $R \to S$ are scalar extensions by Corollary 3.5 and by assumption respectively, $R_\lambda^\wedge \to S$ is faithfully flat by Proposition 3.2, whence injective. □

Corollary 3.7 (*Lifting of Scalar Extensions*)**.** *Let $R \to S$ be a scalar extension with S complete. If R is the homomorphic image of a Noetherian local ring A, then there exists a scalar extension $A \to B$ whose base change is $R \to S$, that is to say, $S = B \otimes_A R$.*

Proof. We leave it to the reader to verify that, after taking completions, we may assume that also A and R are complete. By Cohen's structure theorems, A and R are the homomorphic images of $V[[X]]$ modulo some ideals $J \subseteq I$ respectively, where V is either their common residue field or otherwise a complete p-ring over that residue field, and where X is a finite tuple of indeterminates. Moreover, $S \cong R_\lambda^\wedge$ by Proposition 3.6, where λ is the residue field of S. In particular, $S \cong V_\lambda^\wedge[[X]]/IV_\lambda^\wedge[[X]]$. Hence putting $B := V_\lambda^\wedge[[X]]/JV_\lambda^\wedge[[X]]$ yields a scalar extension $A \to B$ with $A/IA = R \to B/IB = S$, as required. □

The following result is a sharpening of [15, Theorem 2.4].

Corollary 3.8. *Let R be a Noetherian local ring with residue field κ. If $\kappa_\natural$ is the ultrapower of κ, then the catapower $R_\sharp$ of R is equal to the completion $R_{\kappa_\natural}^\wedge$ along $\kappa_\natural$.*

Proof. By Lemma 2.4, the residue field of $R_\sharp$ is $\kappa_\natural$. Since $R \to R_\sharp$ is a scalar extension by Proposition 3.3.3, and since $R_\sharp$ is complete, $R_\sharp \cong R_{\kappa_\natural}^\wedge$ by Proposition 3.6. □

Corollary 3.9. *Let $R \to S$ be a finite local homomorphism inducing a trivial extension on the residue fields. For every extension λ of this common residue field, $S_\lambda^\wedge \cong R_\lambda^\wedge \otimes_R S$.*

Proof. The base change $S \to R_\lambda^\wedge \otimes_R S$ is faithfully flat. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of R and S respectively. Since

$$(R_\lambda^\wedge \otimes_R S)/\mathfrak{n}(R_\lambda^\wedge \otimes_R S) \cong (R_\lambda^\wedge/\mathfrak{m}R_\lambda^\wedge) \otimes_{R/\mathfrak{m}} (S/\mathfrak{n}) \cong \lambda \otimes_\kappa \kappa = \lambda$$

the ideal $\mathfrak{n}(R_\lambda^\wedge \otimes_R S)$ is a maximal ideal. Since the base change $R_\lambda^\wedge \to R_\lambda^\wedge \otimes_R S$ is finite with trivial residue field extension and since $R_\lambda^\wedge$ is complete whence Henselian, $R_\lambda^\wedge \otimes_R S$ is a complete local ring. Hence we showed that $S \to R_\lambda^\wedge \otimes_R S$ is a scalar extension and since the latter ring is complete with residue field equal to λ, it is isomorphic to $S_\lambda^\wedge$ by Proposition 3.6. □

Corollary 3.10. *Suppose R is an excellent local ring. If $R \to S$ is a scalar extension inducing a separable extension on the residue fields, then $R \to S$ is a regular homomorphism.*

Proof. By [14, Theorem 28.10], the scalar extension $R \to S$ is formally smooth, since it is unramified and the residue field extension is separable. The assertion now follows from a result by André in [1] (see also [14, p. 260]). □

In fact, with aid of Proposition 3.6, Corollary 3.7 and Cohen's structure theorems, one reduces to proving that $V[[X]] \to V_\lambda^\wedge[[X]]$ is regular, where V is either a field or a complete p-ring, and where λ is a separable extension of the residue field of V. This approach circumvents the use of André's deep result.

Definition 3.11. A Noetherian local ring R is called *analytically irreducible*, if $\widehat{R}$ is a domain; it is called *absolutely analytically irreducible*, if $R_{\kappa^{\mathrm{alg}}}^\wedge$ is a domain, where κ^{alg} is the algebraic closure of the residue field κ of R; and it is called *universally irreducible*, if any scalar extension of R is a domain.

Corollary 3.12. *If R is an excellent normal local domain with perfect residue field, then R is universally irreducible.*

Proof. Let S be a scalar extension of R. By Corollary 3.10, the map $R \to S$ is regular and hence S is again normal by [14, Theorem 32.2], whence a domain. □

Proposition 3.13. *A Noetherian local ring is absolutely analytically irreducible if and only if it is universally irreducible.*

Proof. Since we will make no essential use of this result, we only give a sketch of a proof. One direction is obvious. For the other, we may reduce to the case that R is a complete Noetherian local domain with algebraically closed residue field κ. We need to show that $R^{\wedge}_{\lambda}$ is a domain, where λ is an arbitrary extension field of κ. By Cohen's structure theorems, there exists a finite extension $S := V[[X]] \subseteq R$, where V is either κ or the complete p-ring over κ, and X is a tuple of indeterminates. Write $R = S[Y]/\mathfrak{p}$ for some finite tuple of indeterminates Y, so that $\mathfrak{p}$ is in particular a prime ideal. Since the fraction field of $S^{\wedge}_{\lambda} = V^{\wedge}_{\lambda}[[X]]$ is a regular extension of the fraction field of $S = V[[X]]$, the same argument as in the proof of [2, Lemma 5.21] then shows that $\mathfrak{p}S^{\wedge}_{\lambda}[Y]$ is a prime ideal. Hence we are done, since $R^{\wedge}_{\lambda} = S^{\wedge}_{\lambda}[Y]/\mathfrak{p}S^{\wedge}_{\lambda}[Y]$ by Corollary 3.9. □

We are ready to formulate a flatness criterion generalizing [13, Theorem 8]; we prove a slightly stronger version than the one quoted in the introduction.

Theorem 3.14. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R is absolutely analytically irreducible, e.g., an excellent normal local domain with perfect residue field, or a complete local domain with algebraically closed residue field. If $R \to S$ is unramified and $\dim(R) = \dim(S)$, then $R \to S$ is faithfully flat, whence a scalar extension.*

Proof. Recall that $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ being unramified means that $\mathfrak{n} = \mathfrak{m}S$. It suffices to prove the assertion under the additional assumption that both R and S are complete. Indeed, if $R \to S$ is arbitrary, then $\widehat{R} \to \widehat{S}$ satisfies again the hypotheses of the theorem and therefore would be faithfully flat. By an easy descent argument, $R \to S$ is then also faithfully flat.

So assume R and S are complete and let λ be the residue field of S. By assumption, $R^{\wedge}_{\lambda}$ is a domain, of the same dimension as R. By the universal property of the completion along λ, we get a local R-algebra homomorphism $R^{\wedge}_{\lambda} \to S$. By [14, Theorem 8.4], this homomorphism is surjective. It is also injective, since $R^{\wedge}_{\lambda}$ and S have the same dimension and $R^{\wedge}_{\lambda}$ is a domain by Proposition 3.13. Hence $R^{\wedge}_{\lambda} \cong S$, so that $R \to S$ is a scalar extension. □

4. Similarity relation

Next, we introduce an equivalence relation on the class of Noetherian local rings, which, although coarser than the isomorphism relation, preserves most local singularity properties (see for instance Theorem 4.1 below). Namely, we say that two Noetherian local rings R and S are *similar*, denoted $R \approx S$, if they admit a common scalar extension. Let T be this common scalar extension. Its completion is again a scalar extension and by Proposition 3.6, it is therefore isomorphic to both $R^{\wedge}_{\lambda}$ and $S^{\wedge}_{\lambda}$, where λ is the residue field of T. In other words, we showed that $R \approx S$ if and only if $R^{\wedge}_{\lambda} \cong S^{\wedge}_{\lambda}$ for some sufficiently large common extension λ of their respective residue fields. It follows easily from this that $\approx$ is an equivalence relation. The collection of all local rings similar to a given Noetherian local ring R is called the *similarity class* of R and is denoted $[R]$. Immediately from the results in [14, Section 23] and [19, Proposition 9.3] (where the notion of a *singularity defect* is introduced), we get:

Theorem 4.1. *If two Noetherian local rings are similar, then they have the same dimension, depth and Hilbert series, and one is regular (respectively, Cohen–Macaulay, Gorenstein, complete intersection) if and only if the other is. More generally, any two similar local rings have the same singularity defects.* □

Using Corollary 3.10, other properties, such as being reduced or normal, are also invariant under the similarity relation, provided the rings are excellent with perfect residue field. Note that being a domain is not preserved under the similarity relation, necessitating Definition 3.11.

Proposition 4.2. *Any two catapowers of a Noetherian local ring are similar, and so are any two Noetherian local rings that are elementary equivalent in the sense of model-theory.*

More generally, let R_w and S_w be sequences of Noetherian local rings of embedding dimension at most d. If almost each R_w is similar to S_w, then the respective cataproducts $R_\natural$ and $S_\natural$ are also similar.

Proof. Suppose R and S are elementary equivalent Noetherian local rings. By the Keisler–Shelah theorem (see [11, Theorem 9.5.7]), some ultrapowers of R and S are isomorphic, whence so are their corresponding catapowers (strictly speaking, the underlying index set will in general no longer be countable, so that we have to make some minor modifications alluded to in footnote 1; details are left to the reader). By Proposition 3.3, these are scalar extensions of R and S respectively, proving the first assertion.

To prove the second assertion, we may without loss of generality assume that all rings are complete. By our discussion above, we may further reduce to the case that S_w is a scalar extension of R_w. Since R_w is a homomorphic image of a d-dimensional regular local ring by Cohen's structure theorems, and since the property we seek to prove is preserved under homomorphic images by Lemma 2.4 and Corollary 3.5, we may moreover assume by Corollary 3.7 that each R_w is regular, of dimension d. By Theorem 4.1, almost all S_w are then also regular of dimension d. By [22, Corollary 5.15], the cataproducts $R_\natural$ and $S_\natural$ are therefore again d-dimensional regular local rings. The induced homomorphism $R_\natural \to S_\natural$ is unramified by Lemma 2.4. Hence, it is faithfully flat by [14, Theorem 23.1], whence a scalar extension, as we wanted to show. □

We denote the collection of all similarity classes of Noetherian local rings by $\mathbb{Sim}$. Although the class of Noetherian local rings is not a set, we no longer have this complication for its quotient:

Proposition 4.3. *The quotient $\mathbb{Sim}$ is a set.*

Proof. Let $[R]$ be a similarity class and let κ be the residue field of R. Since $R \approx \widehat{R}$, we may assume that R is complete, whence, by Cohen's structure theorems, the homomorphic image of $S := V[[X]]$ with V either equal to κ or to the complete p-ring over κ, and with X a finite tuple of indeterminates. Suppose $R = S/I$ with $I = (f_1, \dots, f_s)S$. We may choose a subring W of V of size at most the continuum so that it contains all coefficients of the f_i and so that W is again a field or a complete p-ring. Let $T := W[[X]]$ and $J := (f_1, \dots, f_s)T$, so that $S \cong T^{\wedge}_{\kappa}$ and $I = JS$. Hence, by base change, $R \cong S/I$ is a scalar extension of T/J, showing that $T/J \approx R$. In conclusion, we showed that every similarity class contains a ring of size at most the continuum, and therefore $\mathbb{Sim}$ is a set. □

5. Jet metric

Our next goal is to define a metric on the space $\mathbb{Sim}$. We will first define a semi-metric on the space of all Noetherian local rings. Let $(R, \mathfrak{m})$ be a Noetherian local ring. The *n-th jet* of R (also called the *n-th infinitesimal neighborhood*) is by definition the (Artinian) residue ring $R/\mathfrak{m}^n$ and will be denoted J^nR. Recall that the ($\mathfrak{m}$-adic) completion $\widehat{R}$ of R is the inverse limit of all n-th jets of R, and that $J^nR \cong J^n\widehat{R}$. We define a semi-metric on the class of all Noetherian local rings, called the *jet metric*, as follows. Given two Noetherian local rings R and S, let $d(R, S)$ be the infimum of the numbers 2^{-n} for which $J^nR \cong J^nS$. In words, the distance between two local rings is at most 2^{-n} if their n-th jets agree. One easily verifies that this distance function satisfies all the axioms of a metric, except that two distinct elements can be at distance zero, so that $d(\cdot, \cdot)$ is only a semi-metric. It is an interesting problem to determine all local rings that are d-equivalent to a given local ring; a partial answer is provided in [28]. It is clear that any two Noetherian local rings with the same completion have this property. For our purposes, the following partial solution to this question suffices:

Proposition 5.1. *Given two Noetherian local rings R and S, if $R \sim_d S$, that is to say, if $d(R, S) = 0$, then $R \approx S$.*

Proof. By definition, there exists for each n an Artinian local ring T_n isomorphic to both J^nR and J^nS. Let $R_\natural$ and $R_\sharp$ be the respective ultrapower and catapower of R, and let $T_\natural$ and $T_\sharp$ be the respective ultraproduct and cataproduct of the T_n. Taking ultraproducts of the surjections $R \to T_n$ yields a surjection $R_\natural \to T_\natural$ whence a surjection $R_\sharp \to T_\sharp$. Let $r \in R_\natural$ be an element whose image in $R_\sharp$ lies in the kernel of $R_\sharp \to T_\sharp$, that is to say, $r \in \mathrm{Inf}(T_\natural)$. Let r_n be elements in R with ultraproduct equal to r. Fix some N, and let $\mathfrak{m}$ be the maximal ideal of R. Since $r \in \mathfrak{m}^N T_\natural$, Łoś' Theorem yields $r_n \in \mathfrak{m}^N T_n$ for almost all n. For $n \geq N$, this implies $r_n \in \mathfrak{m}^N$ and hence by Łoś' Theorem, $r \in \mathfrak{m}^N R_\natural$. Since this holds for all N, the image of r in $R_\sharp$ is zero, showing that $R_\sharp \to T_\sharp$ is an isomorphism. Applying the same argument to the catapower $S_\sharp$ of S, we also get $S_\sharp \cong T_\sharp$ and hence $R_\sharp \cong S_\sharp$. Therefore, $R \approx S$ by Proposition 3.3.3. □

The jet semi-metric is non-archimedean, and hence the induced topology, called the *jet topology*, is totally disconnected. By convention, the zero-th jet of a ring is zero (since we think of $\mathfrak{m}^0$ as the unit ideal). It follows that the distance between any two local rings is at most one, that is to say, d is bounded. Immediately from the definitions we also get:

Lemma 5.2. *If $d(R, S) < 1$, then R and S have the same residue field; if $d(R, S) < 1/2$, then R and S have the same embedding dimension.* □

In particular, if, in this metric, R_w is a Cauchy sequence of Noetherian local rings, then almost of all R_w have the same residue field, called the *residue field* of the sequence, and the same embedding dimension. By the above discussion, the cataproduct $R_\sharp$ is therefore a complete Noetherian local ring. By Lemma 5.2, the embedding dimension is a continuous map onto the discrete space $\mathbb{Z}$. This is no longer true for dimension: for instance $R := k[[X]]$ and $R_n := R/X^nR$ lie at distance 2^{-n}, yet their dimensions are not the same. One can show, however, that dimension is upper-semicontinuous.

By an (open) *ball* $\mathbb{B}$ with *center* R and *radius* $0 < \delta \leq 1$, we mean the collection of all Noetherian local rings S such that $d(R, S) < \delta$. Since the metric is non-archimedean, any member of a ball is its center and every ball is both open and closed in the jet topology, that is to say, is a *clopen*. Because the distance function only takes discrete values (the powers of 1/2), any two radii which lie between two consecutive powers of 1/2 yield the same ball. Therefore, by the *radius* of a ball $\mathbb{B}$, we mean twice the largest distance between two members of $\mathbb{B}$; this is always a power of 1/2. (We need to take twice the distance since we used a strict inequality in the definition of a ball.)

A *unit ball* is a ball $\mathbb{B}$ with radius 1 and hence consists of all local rings with the same residue field. We call this common residue field the *residue field* of $\mathbb{B}$. This gives a one-one correspondence between unit balls and fields. More generally, to every ball $\mathbb{B}$, we associate an Artinian local ring $R_{\mathbb{B}}$, called the *residue ring* of $\mathbb{B}$, given as the unique local ring such that $J^nR \cong R_{\mathbb{B}}$, for all $R \in \mathbb{B}$, where 2^{-n+1} is the radius of $\mathbb{B}$. Note that $R_{\mathbb{B}}$ is a center of $\mathbb{B}$ and, moreover, the radius of $\mathbb{B}$ is determined by $R_{\mathbb{B}}$: it is equal to 2^{-n+1} where n is the nilpotency index of $R_{\mathbb{B}}$. In conclusion, there is a one-one correspondence between balls $\mathbb{B}$ and Artinian local rings.

Proposition 5.3. *Every ball is a set.*

Proof. It suffices to prove this for a unit ball $\mathbb{B}$. The result will follow if we can show that there is a cardinal number so that every member of $\mathbb{B}$ has size at most this cardinal. Let κ be the residue field of $\mathbb{B}$ and let $R \in \mathbb{B}$. Since the cardinality of a Noetherian local ring is at most the cardinality of its completion, we may assume that R is complete. By Cohen's structure theorems, R is a homomorphic image of $V[[X]]$, where X is a finite tuple of indeterminates and V is equal to κ in the equal characteristic case, and equal to the complete p-ring over κ in the mixed characteristic case. It is clear that in either case, the cardinality of $V[[X]]$ is bounded in terms of the cardinality of κ, whence so is its homomorphic image R. □

Note that each ball $\mathbb{B}$ is infinite: if $R_{\mathbb{B}}$ is its residue ring, then the latter is of the form S/I, where $(S, \mathfrak{n})$ is a power series ring $V[[X]]$. If n is the nilpotency index of $R_{\mathbb{B}}$, then $S/J \in \mathbb{B}$ for any ideal $J \subseteq S$ such that $J + \mathfrak{n}^n = I$.

Corollary 5.4. *Let $\kappa \subseteq \lambda$ be an extension of fields and let $\mathbb{B}_\kappa$ and $\mathbb{B}_\lambda$ be the unique unit balls with residue field κ and λ, respectively. The map sending a ring in $\mathbb{B}_\kappa$ to its completion along λ is an isometry $\mathbb{B}_\kappa \to \mathbb{B}_\lambda$.*

Proof. Take $R, S \in \mathbb{B}_\kappa$. Clearly, the completions $R^\wedge_\lambda$ and $S^\wedge_\lambda$ along λ belong both to $\mathbb{B}_\lambda$. Suppose $d(R, S) \leq 2^{-n}$, that is to say, their n-th jets $J^n R$ and $J^n S$ are isomorphic. By Corollary 3.5, the completions of $J^n R$ and $J^n S$ along λ are respectively $J^n R^\wedge_\lambda$ and $J^n S^\wedge_\lambda$, and therefore are isomorphic, showing that $d(R^\wedge_\lambda, S^\wedge_\lambda) \leq 2^{-n}$. □

Proposition 5.5. *If $\mathbf{r}$ and $\mathbf{s}$ are Cauchy sequences of Noetherian local rings, say, $\mathbf{r}(w) := R_w$ and $\mathbf{s}(w) := S_w$, with the respective cataproducts $R_\natural$ and $S_\natural$, then $d(R_\natural, S_\natural) \leq d(\mathbf{r}, \mathbf{s})$. In particular, if $\mathbf{r} \sim_d \mathbf{s}$, then $R_\natural \approx S_\natural$.*

Proof. The last assertion is immediate by the first and Proposition 5.1. Suppose $d(\mathbf{r}, \mathbf{s}) \leq 2^{-n}$. This means that for some w_0 and all $w > w_0$, we have $J^n R_w \cong J^n S_w$. By Lemma 2.4, the n-th jets $J^n R_\natural$ and $J^n S_\natural$ are isomorphic, showing that $d(R_\natural, S_\natural) \leq 2^{-n}$. □

The next result shows that cataproducts act as limits up to similarity. To formulate it, we extend our previous notation: let $\mathbf{r}$ be a sequence of Noetherian local rings with the same residue field κ (e.g., a Cauchy sequence) and let λ be an extension field of κ. Then we let $\mathbf{r}^\wedge_\lambda$ denote the sequence of rings obtained by taking the completions along λ of all members of $\mathbf{r}$, that is to say, $\mathbf{r}^\wedge_\lambda(w) := (R_w)^\wedge_\lambda$, if $\mathbf{r}(w) = R_w$.

Theorem 5.6. *Let $\mathbf{r}$ be a Cauchy sequence of Noetherian local rings with residue field κ. Let λ be any extension field of the ultrapower $\kappa_\natural$ of κ. Then $\mathbf{r}^\wedge_\lambda$ is a Cauchy sequence converging to $(R_\natural)^\wedge_\lambda$. In particular, $R_\natural$ is a limit of $\mathbf{r}^\wedge_{\kappa_\natural}$.*

Proof. Let $R_w := \mathbf{r}(w)$. Fix n and choose $w(n)$ so that all $J^n R_w$ for $w \geq w(n)$ are isomorphic, say, to T. By Lemma 2.4, the n-th jet $J^n R_\natural$ is isomorphic to the catapower $T_\natural$; the latter is isomorphic to $T^\wedge_{\kappa_\natural}$ by Corollary 3.8; and this in turn is isomorphic to $J^n\big((R_w)^\wedge_{\kappa_\natural}\big)$, for all $w \geq w(n)$ by Corollary 3.5. In summary, we showed that

$$d((R_w)^\wedge_{\kappa_\natural}, R_\natural) \leq 2^{-n},$$

for all $w \geq w(n)$. By Corollary 5.4, taking completions along λ yields

$$d((R_w)^\wedge_\lambda, (R_\natural)^\wedge_\lambda) \leq 2^{-n},$$

for all $w \geq w(n)$. Since this holds for all n, the assertion follows. □

6. Similarity space

We are ready to define a metric on the similarity space $\mathbb{Sim}$. For two similarity classes $[R]$ and $[S]$, let $d([R], [S])$ be equal to the infimum of all $d(R', S')$ with $R' \approx R$ and $S' \approx S$. Alternatively, recall that for a semi-metric space (Σ, d), the distance $d(U, V)$ between two subclasses U and V is defined to be the infimum of all $d(x, y)$ with $x \in U$ and $y \in V$; hence $d([R], [S])$ is just the distance between $[R]$ and $[S]$ viewed as subclasses. The next result allows us to calculate this distance:

Lemma 6.1. *For any two Noetherian local rings R and S and for any $n \in \mathbb{N}$, we have $d([R], [S]) \leq 2^{-n}$ if and only if $J^n R \approx J^n S$.*

Proof. Suppose $d([R], [S]) \leq 2^{-n}$ and choose $R' \approx R$ and $S' \approx S$ so that $d(R', S') \leq 2^{-n}$. In other words, $J^n R' \cong J^n S'$ and therefore, $J^n R \approx J^n S$ by Corollary 3.5. Conversely, assume $J^n R \approx J^n S$ and let T be a common scalar extension of $J^n R$ and $J^n S$. Let λ be the residue field of T. By Corollary 3.5, the n-th jets of $R^\wedge_\lambda$ and $S^\wedge_\lambda$ are equal to T. In other words, $d(R^\wedge_\lambda, S^\wedge_\lambda) \leq 2^{-n}$. Since $d([R], [S])$ is defined as an infimum, it is at most 2^{-n}. □

Corollary 6.2. *The quotient $(\mathbb{Sim}, d)$ is a metric space.*

Proof. Suppose $d([R], [S]) = 0$. By Lemma 6.1, the n-th jets $J^n R$ and $J^n S$ of R and S are similar for all n. Hence there exists a common scalar extension T_n of $J^n R$ and $J^n S$. We may inductively choose T_{n+1} to have a residue field containing the residue field of T_n by Corollary 5.4, since scalar extensions can only make the distance smaller. Let λ be the union of all these residue fields. By Corollary 3.5, the n-th jets of $R^\wedge_\lambda$ and $S^\wedge_\lambda$ are equal to $(T_n)^\wedge_\lambda$. Since this holds for all n, we showed that $d(R^\wedge_\lambda, S^\wedge_\lambda) = 0$. By Proposition 5.1, we get $R^\wedge_\lambda \approx S^\wedge_\lambda$ and hence $[R] = [R^\wedge_\lambda] = [S^\wedge_\lambda] = [S]$. □

It follows from Theorem 5.6 that given a Cauchy sequence $\mathbf{r}$ of Noetherian local rings, the sequence $\mathbf{r}^\wedge_{\kappa_\natural}$ has a limit, where $\kappa_\natural$ is the ultrapower of the residue field of $\mathbf{r}$. Since the corresponding members of $\mathbf{r}$ and $\mathbf{r}^\wedge_{\kappa_\natural}$ are similar, we showed that every Cauchy sequence becomes convergent after replacing each of its components by an appropriately chosen similar ring. Therefore, the next result should not come as a surprise:

Theorem 6.3. *The metric space $\mathbb{Sim}$ is complete.*

Proof. To define an isometry $\widehat{\imath}\colon \widehat{\mathbb{Sim}} \to \mathbb{Sim}$, we start with defining a map $i\colon \operatorname{Cau}(\mathbb{Sim}) \to \mathbb{Sim}$. Let $\mathbf{r}$ be a Cauchy sequence in $\mathbb{Sim}$. For each w, let R_w be a representative in the similarity class $\mathbf{r}(w)$, and let $R_\sharp$ be their cataproduct. Note that $R_\sharp$ is a complete Noetherian local ring since almost all R_w have the same embedding dimension. Define $i(\mathbf{r}) := [R_\sharp]$. By Proposition 4.2, the map i is well-defined, that is to say, does not depend on the choice of representatives R_w. Suppose $\mathbf{s}$ is a second Cauchy sequence which is equivalent to $\mathbf{r}$ and let $S_\sharp$ be the cataproduct of the representatives S_w of each $\mathbf{s}(w)$. For a fixed n, we have $d([R_w], [S_w]) \le 2^{-n}$ for all sufficiently large w. By Lemma 6.1, the n-th jets of R_w and S_w are therefore similar, for all sufficiently large w. By Proposition 4.2, then so are the n-th jets of $R_\sharp$ and $S_\sharp$, so that $d([R_\sharp], [S_\sharp]) \le 2^{-n}$ by another application of Lemma 6.1. Since this holds for all n, Corollary 6.2 yields $[R_\sharp] = [S_\sharp]$. By definition of completion, i therefore factors through a map

$$\widehat{\imath}\colon \widehat{\mathbb{Sim}} \to \mathbb{Sim}.$$

We leave it to the reader to check that $\widehat{\imath}$ preserves the metric. Note that $\widehat{\imath}$ restricted to $\mathbb{Sim}$ is the identity, since a catapower is a scalar extension by Proposition 4.2. Hence $\widehat{\imath}$ must be surjective. To prove injectivity, assume that $\mathbf{r}$ and $\mathbf{s}$ are Cauchy sequences of Noetherian local rings whose respective cataproducts $R_\sharp$ and $S_\sharp$ are similar. Let λ be a large enough field extension so that

$$(R_\sharp)^\wedge_\lambda \cong (S_\sharp)^\wedge_\lambda.$$

By Theorem 5.6, the (component-wise) completion $\mathbf{r}^\wedge_\lambda$ along λ converges to $(R_\sharp)^\wedge_\lambda$, and likewise $\mathbf{s}^\wedge_\lambda$ converges to $(S_\sharp)^\wedge_\lambda$. Therefore, $\mathbf{r}^\wedge_\lambda$ and $\mathbf{s}^\wedge_\lambda$, as they converge to the same limit, are equivalent, which proves that $\widehat{\imath}$ is injective. □

We have the following generalization of Proposition 4.2.

Corollary 6.4. *If R_w is a Cauchy sequence, then any two cataproducts of R_w (with respect to different ultrafilters) are similar. In particular, if the common residue field κ of the R_w is an algebraically closed field, then the cataproduct of the R_w is, up to isomorphism, independent from the choice of ultrafilter.*

Proof. According to the proof of Theorem 6.3, the similarity class of any cataproduct $R_\sharp$ of the R_w is a limit of the sequence of similarity classes $[R_w]$, and therefore, is unique by Corollary 6.2. Since any two ultrapowers of κ are algebraically closed and have the same (uncountable) cardinality, they are isomorphic by Leibnitz's theorem. Since any two cataproducts of the R_w are similar by the first assertion, and are complete with isomorphic residue fields, they must be isomorphic by Proposition 3.6. □

We introduce the following notation. Let $\mathbb{S} \subseteq \mathbb{Sim}$ be a subset, and let $d \ge 0$ and $e \ge 1$. We let $\mathbb{S}_d$ (respectively, $\mathbb{S}_{d,e}$) be the set of similarity classes of Noetherian local rings in $\mathbb{S}$ having dimension d (and parameter degree e). Recall that the *parameter degree* of R is the minimal length of a residue ring R/I, where I runs over all parameter ideals of R. It is not hard to show that two similar rings with infinite residue field have the same parameter degree, and so we may speak of the parameter degree of a similarity class as the parameter degree of any of its members having infinite residue field.

Corollary 6.5. *For each $d \ge 0$ and $e \ge 1$, the subset $\mathbb{Sim}_{d,e} \subseteq \mathbb{Sim}$ is closed.*

Proof. It suffices to show that $\mathbb{Sim}_{d,e}$ is closed under limits. Hence let $\mathbf{r}$ be a Cauchy sequence in $\mathbb{Sim}_{d,e}$, and choose representatives R_w in each $\mathbf{r}(w)$, of dimension d and parameter degree e. Let $R_\sharp$ be the cataproduct of the R_w, so that its similarity class is the limit of $\mathbf{r}$ by Theorem 6.3. Since the cataproduct $R_\sharp$ has dimension d and parameter degree e by [22, Theorem 5.22], the claim follows. □

We can now state and prove the main theorem of this paper:

Theorem 6.6. *The metric space $\mathbb{Sim}$ is a Polish space. In particular, the similarity relation is smooth.*

Proof. In view of Theorem 6.3, it remains to show that $\mathbb{Sim}$ contains a countable dense subset. We already observed that there is a one-one correspondence between balls and Artinian local rings, so that $\mathbb{Sim}_0$, the similarity classes of Artinian local rings, form a dense subset of $\mathbb{Sim}$. Let R be an Artinian local ring with residue field κ. By Cohen's structure theorems, R is of the form $V[[X]]/I$, where V is either κ or the complete p-ring over κ, and where X is a tuple of indeterminates. Since R is Artinian, it is in fact finitely generated over V. Hence, by an argument similar to the one in the proof of Proposition 4.3, there exists a finitely generated subfield $\kappa_0 \subseteq \kappa$ and an Artinian local ring R_0 with residue field κ_0, such that $R_0 \approx R$. Since there are only countably many finitely generated fields, the collection of all these R_0 is again countable. □

7. Variants

A first variant is obtained by working instead in the category of Noetherian local Z-algebras, for Z some Noetherian ring, where the morphisms are now given by local Z-algebra homomorphisms. This leads to the notion of two Z-algebras being *Z-similar*, and the same argument shows that, provided Z is countable, classifying Noetherian local Z-algebras up to Z-similarity is again a smooth problem.

We may also extend the definition to include modules. Namely, given an R-module M and an S-module N, we say that $d(M, N) \le 2^{-n}$, if J^nR and J^nS have a common scalar extension T such that $M \otimes_R T \cong N \otimes_S T$. In particular, $d(R, S) \le d(M, N)$. We will not study the similarity problem for modules—and at present, I do not know whether this is a smooth classification problem, even over a fixed ring. We will use this metric in the proof of Theorem 8.3; see also [22, Section 11] for some further applications.

We now turn to some other classification problems that can be reduced to classification up to similarity.

7.1. Classification of analytic germs

Let κ be a field. By an *analytic germ* over κ, we mean a complete Noetherian local ring with residue field κ; we denote the set of isomorphism classes of analytic germs by $\mathbb{I}\mathrm{so}\mathbb{A}\mathrm{n}(\kappa)$. It should be stressed that we do not assume that the isomorphism is defined over κ. Note that if κ has prime characteristic p, then the germ can either have equal or mixed characteristic. By the Cohen structure theorem, analytic germs are simply homomorphic images of power series rings $V[[X]]$, with V either κ (equal characteristic) or the complete p-ring over κ (mixed characteristic). Assume, moreover, that κ is algebraically closed and has size of the continuum. It follows that every (countable) ultrapower $\kappa_\natural$ of κ is again algebraically closed and has the same cardinality as κ, whence by Leibnitz's theorem, is isomorphic with κ. In the mixed characteristic case, by uniqueness of p-rings, the catapower of V is then also isomorphic to V. This shows that the set of analytic germs over such a field κ is, up to isomorphism, closed under cataproducts, whence under limits. Moreover, there are, up to isomorphism, only countably many analytic germs of dimension zero, and they form a dense subset $\mathbb{I}\mathrm{so}\mathbb{A}\mathrm{n}_0(\kappa)$ of $\mathbb{I}\mathrm{so}\mathbb{A}\mathrm{n}(\kappa)$. In conclusion, we showed Theorem 1.1 from the introduction. Note that in the above, we may replace the size of the continuum by any cardinal of the form 2^γ, with γ an infinite cardinal. In fact, under the Generalized Continuum Hypothesis, this means *any* uncountable cardinal.

7.2. Classification of infinitesimal deformations

By a *deformator* $\mathbf{R}$, we mean a pair $(R, \mathbf{x})$, with $(R, \mathfrak{m})$ a Noetherian local ring and $\mathbf{x} := (x_1, \ldots, x_d)$ a tuple generating an $\mathfrak{m}$-primary ideal. To emphasize the maximal ideal, we may also represent the deformator $\mathbf{R}$ as the triple $(R, \mathfrak{m}, \mathbf{x})$. We call R and $\mathbf{x}$ respectively the *underlying ring* and *tuple* of $\mathbf{R}$, and we call the length of $R/\mathbf{x}R$ the *colength* of $\mathbf{R}$. We call $\mathbf{R}$ *parametric*, if $\mathbf{x}$ is a system of parameters. When we say that a deformator has a certain ring theoretic property, then we mean that its underlying ring has this property. Let $\mathbf{S} := (S, \mathbf{y})$ be a second deformator, with $\mathbf{y} = (y_1, \ldots, y_e)$. A *morphism* $\mathbf{R} \to \mathbf{S}$ *of deformators*, is a ring homomorphism $R \to S$ mapping $\mathbf{x}$ to $\mathbf{y}$. In particular, there are no morphisms between deformators with tuples of different length. It is easy to verify that these definitions make the class of deformators into a category. We call a morphism $\mathbf{R} \to \mathbf{S}$ *flat, unramified, a scalar extension, etc.*, if and only if the underlying homomorphism $R \to S$ has this property. We say that $\mathbf{R}$ and $\mathbf{S}$ are *similar*, in symbols, $\mathbf{R} \approx \mathbf{S}$, if they have a common scalar extension $\mathbf{T}$ (as deformators). As before, we denote the similarity class of a deformator $\mathbf{R}$ by $[\mathbf{R}]$.

The *n-th infinitesimal deformation* of a deformator $\mathbf{R} := (R, \mathbf{x})$, denoted $J^n\mathbf{R}$, is by definition the Artinian deformator $(R/\mathbf{x}^{(n)}R, \mathbf{x})$, where for an arbitrary tuple $\mathbf{y} := (y_1, \ldots, y_s)$, we write $\mathbf{y}^{(n)}$ for the tuple $(y_1^n, \ldots, y_s^n)$. If $\mathbf{R} \to \mathbf{S}$ is a morphism of deformators, then it induces, for each n, a morphism $J^n\mathbf{R} \to J^n\mathbf{S}$.

Lemma 7.3. *If* $\mathbf{R}$ *and* $\mathbf{S}$ *are similar deformators, then* $J^n\mathbf{R} \approx J^n\mathbf{S}$*, for all n.*

Proof. Since the respective underlying rings R and S are similar, they have the same dimension. Without loss of generality, we may assume that $\mathbf{R} \to \mathbf{S}$ is a scalar extension. By definition of morphism, under the scalar extension $R \to S$, the tuple of $\mathbf{R}$ is sent to that of $\mathbf{S}$, and the assertion is now clear. □

Let $\mathbb{S}\mathrm{im}\mathbb{D}\mathrm{ef}$ denote the set of similarity classes of deformators (the argument that this is indeed a set is analogous to the one for $\mathbb{S}\mathrm{im}$). We define the *deformation metric* on $\mathbb{S}\mathrm{im}\mathbb{D}\mathrm{ef}$ in analogy with the jet metric: given two similarity classes of deformators $[\mathbf{R} := (R, \mathbf{x})]$ and $[\mathbf{S} := (S, \mathbf{y})]$, we set $d([\mathbf{R}], [\mathbf{S}]) \le 2^{-n}$, if $J^n\mathbf{R} \approx J^n\mathbf{S}$. By Lemmas 6.1 and 7.3, this definition is independent from the choice of representatives. Moreover, if $J^n\mathbf{R} \approx J^n\mathbf{S}$, then the definition of morphisms in the category of deformators implies that $J^i\mathbf{R} \approx J^i\mathbf{S}$, for all $i \le n$. Indeed, we may reduce to the case that we have a scalar extension $J^n\mathbf{R} \to J^n\mathbf{S}$, which therefore maps $\mathbf{x}$ to $\mathbf{y}$, and the claim is now clear. If $d(\mathbf{R}, \mathbf{S}) < 1$, then $\mathbf{R}$ and $\mathbf{S}$ have in particular the same colength. As with rings, we will often identify a similarity class with any deformator contained in it, and so we will omit brackets in our notation and speak of the distance between deformators. The connection between the jet metric and the deformational metric is given by:

Proposition 7.4. *If* $\mathbf{R}$ *and* $\mathbf{S}$ *are deformators with respective underlying rings R and S, then* $d(R, S) \le d(\mathbf{R}, \mathbf{S})$*. Conversely, for every deformator* $\mathbf{R}$*, if T is a Noetherian local ring at distance* ε *from R, then we can find a deformator* $\mathbf{T}$ *with underlying ring T, such that* $d(\mathbf{R}, \mathbf{T}) \le \varepsilon^{1/(lm+1)}$*, where l is the colength of* $\mathbf{R}$ *and m the length of its tuple. If, moreover,* $\mathbf{R}$ *is parametric, and* $\dim(R) = \dim(T)$*, then we may also choose* $\mathbf{T}$ *to be parametric.*

Proof. Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be the respective underlying rings of $\mathbf{R}$ and $\mathbf{S}$, and let $\mathbf{x}$ and $\mathbf{y}$ be their respective tuples. If $J^k\mathbf{R} \approx J^k\mathbf{S}$, for some k, then clearly $J^kR \approx J^kS$, since $\mathbf{x}^{(k)}R \subseteq \mathfrak{m}^k$ and $\mathbf{y}^{(k)}S \subseteq \mathfrak{n}^k$. This proves the first assertion.

To prove the second, observe that since $\mathfrak{m}^l \subseteq I := \mathbf{x}R$, we get

$$\mathfrak{m}^{lmn} \subseteq I^{mn} \subseteq \mathbf{x}^{(n)}R, \tag{6}$$

for all n. Hence $\bar{R} := R/\mathbf{x}^{(n)}R$ is a homomorphic image of $J^{lmn}R$. Suppose $d(R, T) \le 2^{-k}$, so that $J^kR \approx J^kT$. Without loss of generality, we may assume that $J^kR \to J^kT$ is a scalar extension. Let $\mathbf{z}$ be a lifting in T of the image of $\mathbf{x}$ in J^kT under this scalar extension, and put $\mathbf{T} := (T, \mathbf{z})$. Let n be an integer strictly less than k/lm, so that $lmn < k$. We want to show that $\mathfrak{p}^k \subseteq \mathbf{z}^{(n)}T$, where $\mathfrak{p}$ is the maximal ideal of T. Put $\bar{T} := T/\mathbf{z}^{(n)}T$. The map $J^kR \to J^kT$ induces a scalar extension $\bar{R} \to \bar{T}/\mathfrak{p}^k\bar{T}$. By (6), the latter is annihilated by $\mathfrak{p}^{lmn}$. Hence $\mathfrak{p}^{lmn}\bar{T} = \mathfrak{p}^k\bar{T}$, and since $lmn < k$, Nakayama's Lemma yields $\mathfrak{p}^{lmn}\bar{T} = 0$, and the claim follows. In particular, base change induces a scalar extension $\bar{R} \to \bar{T}$, and hence a scalar extension $J^n\mathbf{R} \to J^n\mathbf{T}$ of deformators, showing that $d(\mathbf{R}, \mathbf{T}) \le 2^{-n}$, as we wanted to show. □

Theorem 7.5. *Classification of deformators up to similarity is smooth, or, more precisely,* $\mathbb{SimDef}$ *is a Polish space.*

Proof. Let $\mathbf{R}_w$ be a Cauchy sequence in $\mathbb{SimDef}$, and let R_w be the corresponding sequence of underlying rings. By Proposition 7.4, this latter sequence is also Cauchy, whence has a limit in $\mathbb{Sim}$ by Theorem 6.3. In fact, we may take the cataproduct $R_\natural$ of the R_w as a representative of this limit. Since all tuples in $\mathbf{R}_w$ must have the same length, their ultraproduct yields a finite tuple in $\mathbf{x}$ in $R_\natural$. Moreover, almost all $\mathbf{R}_w$ have the same colength, which, by Łoś' Theorem, is then also the length of $R_\natural/\mathbf{x}R_\natural$. In particular, $(R_\natural, \mathbf{x})$ is a deformator. The second part of Proposition 7.4 shows that it is the limit of the $\mathbf{R}_w$. This proves that $\mathbb{SimDef}$ is complete. It remains to show that the subset $\mathbb{SimDef}_0$ of Artinian deformators is countable and dense. However, we argued in the proof of Theorem 6.6 that each similarity class of an Artinian local ring R contains a representative R_0 with a finitely generated residue field. Given any (finite) tuple $\mathbf{x}$, we may choose R_0 so that it also contains $\mathbf{x}$. From this it is easy to see that $\mathbb{SimDef}_0$ is countable, and density is also immediate. □

We denote the subset of similarity classes of parametric deformators by $\mathbb{SimPar}$. Dimension, as this is equal to the length of the tuple, partitions this space in the pieces $\mathbb{SimPar}_d$. It follows immediately from the above proof that each $\mathbb{SimPar}_d$ is a complete subspace of $\mathbb{SimDef}$. In particular, $\mathbb{Sim}_0$ is isometric with $\mathbb{SimPar}_0$. However, for $d > 0$, it is no longer clear whether $\mathbb{SimPar}_d$ has a countable dense subset, and therefore, it might fail to be a Polish subspace.

7.6. Classification of polarized schemes up to isomorphism

Our next application is to the classification of projective schemes. We will tacitly assume that a *projective scheme X* is always of finite type over some field κ. A *polarization* of X is a choice of a very ample line bundle $\mathcal{L}$ on X; we refer to this situation also by calling $\mathfrak{X} := (X, \mathcal{L})$ a *polarized scheme over* κ, and we say that X is the *underlying projective scheme* of $\mathfrak{X}$. In particular, a polarization $(X, \mathcal{L})$ corresponds to a closed immersion $i\colon X \to \mathbb{P}^n_\kappa$, for some n, such that $\mathcal{L} \cong i^*\mathcal{O}(1)$, where $\mathcal{O}(1)$ is the canonical twisting sheaf on $\mathbb{P}^n_\kappa$.

The *section* ring of a polarized scheme $\mathfrak{X} := (X, \mathcal{L})$ is defined as the graded κ-algebra

$$S(\mathfrak{X}) := \sum_{n=0}^{\infty} H^0(X, \mathcal{L}^n).$$

Note that, since $\mathcal{L}$ is very ample, $S(\mathfrak{X})$ is a *standard graded* κ*-algebra*, meaning that it has no homogeneous components of negative degree, its degree zero component is κ, and, as an algebra over κ, it is generated by its homogeneous elements of degree one.

The *vertex algebra* of $\mathfrak{X}$ is the localization of $S(\mathfrak{X})$ at the irrelevant ideal of all elements of positive degree, and will be denoted by $\mathrm{Vert}(\mathfrak{X})$. If X is irreducible and reduced, then the field of fractions of $S(\mathfrak{X})$ (and hence of $\mathrm{Vert}(\mathfrak{X})$) is equal to the function field $\kappa(X)$. In particular, $\mathrm{Vert}(\mathfrak{X})$ is a birational invariant of X. In fact, more is true: the polarized scheme $\mathfrak{X} := (X, \mathcal{L})$ can be recovered from its section ring $S := S(\mathfrak{X})$ as $X = \mathrm{Proj}(S)$ and $\mathcal{L} = \widetilde{S(1)}$, where $S(1)$ is the *Serre twist* of S. We therefore say that two polarized schemes $\mathfrak{X} := (X, \mathcal{L})$ and $\mathfrak{Y} := (Y, \mathcal{M})$ are *isomorphic*, if their section rings are isomorphic as graded algebras, and this is then equivalent with the existence of an isomorphism $f\colon X \to Y$ of projective schemes, such that $f^*\mathcal{M} = \mathcal{L}$.

Let $\mathbb{IsoPol}_\kappa$ be the set of isomorphism classes of polarized schemes over κ. We metrize this space via pull-back along the vertex functor, that is to say,

$$d(\mathfrak{X}, \mathfrak{Y}) := d(\mathrm{Vert}(\mathfrak{X}), \mathrm{Vert}(\mathfrak{Y})).$$

The following easy lemma allows us to calculate this distance function:

Lemma 7.7. *Let* $\mathfrak{X} := (X, \mathcal{L})$ *be a polarized scheme over* κ *with vertex algebra* $R := \mathrm{Vert}(\mathfrak{X})$. *For each n, we have an isomorphism of graded Artinian* κ*-algebras*

$$J^nR \cong \bigoplus_{i=0}^{n-1} H^0(X, \mathcal{L}^i).$$

Proof. Let $S := S(\mathfrak{X})$ be the section ring of $\mathfrak{X}$, and let $\mathfrak{m}$ be the irrelevant maximal ideal. Since $R = S_\mathfrak{m}$, we have $J^nR = S/\mathfrak{m}^n$, for all n. Since S is a standard graded algebra, $\mathfrak{m}^n$ consists of all elements of degree at least n, that is to say,

$$\mathfrak{m}^n = \bigoplus_{i\geq n} H^0(X, \mathcal{L}^i),$$

from which the assertion follows immediately. □

We can now show that we have indeed a metric on $\mathbb{IsoPol}_\kappa$:

Corollary 7.8. *If two polarized schemes* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *over* κ *are at distance zero, then they are isomorphic.*

Proof. By Lemma 7.7, their section rings are isomorphic, and we already argued that this means that the two polarized schemes are isomorphic. □

Proof of Theorem 1.2. We will show that $\mathbb{I}\mathrm{so}\mathbb{P}\mathrm{ol}_\kappa$ is a Polish space, and to this end, we need to show that it contains a countable dense subset and is closed under limits. By Lemma 7.7, the polarizations $(X, \mathcal{L})$ of zero-dimensional projective schemes are dense. Any such scheme is the base change of a zero-dimensional projective scheme X_0 over a finitely generated field, and since very ample line bundles are generated by their global sections, we may choose X_0 so that it admits a very ample line bundle $\mathcal{L}_0$ which induces the line bundle $\mathcal{L}$ on X by base change. This shows that, up to isomorphism, there are only countably many polarizations of zero-dimensional projective schemes.

So remains to show that every Cauchy sequence $\mathfrak{X}_w := (X_w, \mathcal{L}_w)$ in $\mathbb{I}\mathrm{so}\mathbb{P}\mathrm{ol}_\kappa$ has a limit. Let $R_w := \mathrm{Vert}(\mathfrak{X}_w)$, so that by definition, R_w is a Cauchy sequence of Noetherian local rings. Let $R_\sharp$ be the cataproduct of the R_w. By the same argument as in the proof of Theorem 1.1, our assumption on the field κ implies that $R_\sharp$ has residue field isomorphic to κ. Let R be an isomorphic copy of $R_\sharp$ having residue field κ, and let $f : R_\sharp \to R$ be the corresponding isomorphism. Fix some n. By Lemma 7.7, $J^nR \cong J^nR_w$, for all $w \gg 0$. In particular, the n-th homogeneous piece $S_n := H^0(X_w, \mathcal{L}_w^n)$ is independent from w, for w sufficiently large. Since S_n has finite length, its ultrapower is equal to its catapower, and, therefore, via f, isomorphic to itself. Let $S := \oplus_n S_n$. One verifies that this is a standard graded κ-algebra. For instance, to define the ring structure on S, it suffices to define the multiplication of two homogeneous elements, say $a \in S_i$ and $b \in S_j$. Take w large enough so that $H^0(X_w, \mathcal{L}_w^{i+j})$ is equal to S_{i+j}. Choose $a_w, b_w \in S(\mathfrak{X}_w)$ so that their images in R_w have ultraproducts $a_\natural, b_\natural \in R_\natural$ with $f(a_\natural) = a$ and $f(b_\natural) = b$. By Łoś' Theorem, a_w and b_w are homogeneous of degree i and j respectively. We then define ab as the image under f of the ultraproduct of the $a_w b_w \in S_{i+j}$. The other properties are checked similarly. In particular, $J^nR \cong S_0 \oplus \cdots \oplus S_n$, showing that R is the localization of S at its irrelevant maximal ideal. Let $\mathfrak{X} := (X, \mathcal{L})$ be the polarized scheme determined by S, namely, let $X := \mathrm{Proj}(S)$ and $\mathcal{L} := \widetilde{S(1)}$. Hence, it remains to show that $\mathfrak{X}$ is the limit of the $\mathfrak{X}_w$, and this is immediate from the fact that $\mathrm{Vert}(\mathfrak{X}) = R$. □

8. Prolegomena to a complete set of invariants: slopes

Theorem 6.6, although promising, is far from an efficient classification up to similarity. In this final section, we will discuss some (albeit feeble) attempts to make it more concrete. As mentioned in the introduction, any two (uncountable) Polish spaces are Borel equivalent, namely to the standard Borel space on the reals. So, given any (concrete) Polish space $\mathbb{B}$, we ask for a Borel bijection $q : \mathbb{S}\mathrm{im} \to \mathbb{B}$.

8.1. Slopes

Let us call a map $q : \mathbb{S}\mathrm{im} \to \mathbb{B}$ a *pre-slope*, if it is continuous, and a *slope*, if it is moreover injective. Of course, the identity map into $\mathbb{S}\mathrm{im}$ itself is a slope, but we seek more concrete examples. A solution to the classification problem would, for instance, be provided by any real-valued slope. Extending this terminology, let us say that for some subset $\mathbb{S} \subseteq \mathbb{S}\mathrm{im}$ and a map $q : \mathbb{S}\mathrm{im} \to \mathbb{B}$, that q is a *pre-slope* on $\mathbb{S}$ when its restriction to $\mathbb{S}$ is continuous, and a *slope* on $\mathbb{S}$, if it is moreover injective. A priori, the theory only predicts that we can find a real-valued, injective Borel map, which in general is only continuous outside a meagre subset, but perhaps we may venture to postulate the existence of a countable partition $\{\mathbb{S}_i\}$ of $\mathbb{S}\mathrm{im}$, and a map $q : \mathbb{S}\mathrm{im} \to \mathbb{R}$, such that q restricted to each piece $\mathbb{S}_i$ is a slope. Moreover, we want this partition to be indexed by some natural discrete invariants that are preserved under the similarity relation, like dimension and/or parameter degree. We start with some examples of pre-slopes taking values into a concrete complete Polish space (from now on, we will confuse a similarity class with any of its members):

Proposition 8.2. *Viewing $\mathbb{Z}[[t]]$ as a Polish space via its t-adic metric, the map* $\mathrm{Hilb} : \mathbb{S}\mathrm{im} \to \mathbb{Z}[[t]]$ *induced by associating to a Noetherian local ring its Hilbert series* $\mathrm{Hilb}(R)$*, is a pre-slope.*

Proof. Recall that the *Hilbert series* of $(R, \mathfrak{m})$ is defined to be the formal power series

$$\mathrm{Hilb}(R) := \sum_{n=0}^{\infty} \ell(\mathfrak{m}^n/\mathfrak{m}^{n+1})t^n.$$

If R and S are similar, then they have the same Hilbert series, showing that Hilb is defined on $\mathbb{S}\mathrm{im}$. By an easy calculation, $\ell(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \ell(J^{n+1}R) - \ell(J^nR)$. Hence, if R_w converges to R, then for each n, we have $J^nR = J^nR_w$, for all sufficiently large w, showing that Hilb is continuous. □

For a second example of a pre-slope, we make the following definition. Let R be a local ring with residue field κ, and let M be a finitely generated R-module. The *n-th Betti number* $\beta_n(M)$ of M is defined as the vector space dimension of $\mathrm{Tor}_n^R(M, \kappa)$. Alternatively, at least in the Noetherian case, the Betti numbers are the ranks in a minimal free resolution of M, and, hence by Nakayama's Lemma, the minimal number of generators of the syzygies of M. The generating series of these Betti numbers, that is to say, the formal power series

$$\mathrm{Poin}(M) := \sum_n \beta_n(M)t^n$$

is called the *Poincare series* of M. We define the *residual Poincare series* of R to be the Poincare series of its residue field, and denote it $\mathrm{Poin}^{\mathrm{res}}(R)$. If $R \to S$ is a scalar extension, and $F_\bullet$ a minimal free resolution of the residue field κ of R, then by flatness,

$F_\bullet \otimes_R S$ is a minimal free resolution of $\kappa \otimes_R S$, and the latter is the residue field of S, since $R \to S$ is unramified. Hence, any two similar rings have the same residual Poincare series. Let $\mathbb{S}\mathrm{im}\mathbb{C}\mathbb{M}$ be the subset of $\mathbb{S}\mathrm{im}$ consisting of all similarity classes of local Cohen–Macaulay rings. By [22, Corollary 8.7], if we also fix dimension and multiplicity, then each $\mathbb{S}\mathrm{im}\mathbb{C}\mathbb{M}_{d,e}$ is closed under limits.

Theorem 8.3. *The residual Poincare series is a pre-slope on each* $\mathbb{S}\mathrm{im}\mathbb{C}\mathbb{M}_{d,e}$.

Proof. The continuity of the map associating to a d-dimensional local Cohen–Macaulay ring R of multiplicity e its residual Poincare series $\mathrm{Poin}^{\mathrm{res}}(R)$ is an immediate consequence of [22, Theorem 11.4]. Indeed, if R_w is a Cauchy sequence, then, for any fixed n, the residue field of each R_w has the same n-th Betti number, for w sufficiently large, by the cited result. By [22, Proposition 8.9], this is then also the Betti number of the cataproduct $R_\sharp$, that is to say, up to similarity, the limit of the R_w. Therefore, $\mathrm{Poin}^{\mathrm{res}}(R_w)$ converges, in the t-adic topology, to $\mathrm{Poin}^{\mathrm{res}}(R_\sharp)$. □

For a local Cohen–Macaulay ring R, we define its *canonical Poincare series*, denoted $\mathrm{Poin}^{\mathrm{can}}(R)$, as the Poincare series $\mathrm{Poin}(\omega_{\widehat{R}})$ of the canonical module $\omega_{\widehat{R}}$ of its completion $\widehat{R}$ (note that the canonical module always exists when the ring is complete; see for instance [3, Section 3.3]). In particular, R is Gorenstein if and only if its canonical Poincare series is constant (equal to 1): indeed, R is Gorenstein if and only if $\omega_{\widehat{R}} \cong \widehat{R}$. It is not hard to check that the canonical Poincare series is independent from the choice of representative of a similarity class of a Cohen–Macaulay local ring (by the same argument as in [3, Theorem 3.3.5(c)]). Let R and S be two rings in $\mathbb{S}\mathrm{im}\mathbb{C}\mathbb{M}_{d,e}$ at distance at most 2^{-de-1}. After a scalar extension, we may assume that R and S are complete, with infinite residue fields κ and λ, respectively. In particular, there exists a system of parameters $\mathbf{x}$ in R such that $\bar{R} := R/\mathbf{x}R$ has length e. Proposition 7.4 then yields a system of parameters $\mathbf{y}$ in S such that $\bar{R} \approx \bar{S} := S/\mathbf{y}S$. Since the canonical module ω_R is a maximal Cohen–Macaulay, $\mathbf{x}$ is ω_R-regular, and likewise, $\mathbf{y}$ is ω_S-regular. Therefore,

$$\begin{aligned}\mathrm{Tor}_n^R(\omega_R, \kappa) &\cong \mathrm{Tor}_n^{\bar{R}}(\omega_R/\mathbf{x}\omega_R, \kappa)\\ \mathrm{Tor}_n^S(\omega_S, \lambda) &\cong \mathrm{Tor}_n^{\bar{S}}(\omega_S/\mathbf{y}\omega_S, \lambda)\end{aligned} \tag{7}$$

for all n. Since $\bar{R}$ and $\bar{S}$ are similar, they have the same canonical Poincare series $P(t)$. By [3, Theorem 3.3.5], the respective canonical modules of $\bar{R}$ and $\bar{S}$ are $\omega_R/\mathbf{x}\omega_R$ and $\omega_S/\mathbf{y}\omega_S$. By (7), therefore, the canonical Poincare series of R and S are both equal to $P(t)$. In conclusion, we showed:

Proposition 8.4. *On each ball of radius* 2^{-de-1} *in* $\mathbb{S}\mathrm{im}\mathbb{C}\mathbb{M}_{d,e}$, *the canonical Poincare series is constant. In particular, if one of its members is Gorenstein, then so is any.* □

Since any ball is open, we showed:

Corollary 8.5. *The subset of* $\mathbb{S}\mathrm{im}\mathbb{C}\mathbb{M}_{d,e}$ *consisting of the classes of all Gorenstein rings is a clopen.* □

8.6. Quasi-slopes

To find a slope, it is enough to have it defined on the countable dense open subset $\mathbb{S}\mathrm{im}_0$ given by Theorem 6.6. For any map $q_0 : \mathbb{S}\mathrm{im}_0 \to \mathbb{B}$ into a complete metric space (not necessarily continuous), define its extension $\widehat{q}_0$ as the partial map $\mathbb{S}\mathrm{im} \dashrightarrow \mathbb{B}$ given as the limit of the $q_0(J^nR)$, for R a Noetherian local ring, whenever this limit exists. Note that if $R \approx S$, then $q_0(J^nR)$ converges if and only if $q_0(J^nS)$ does, and their limits are similar. In particular, $q_0(R) = \widehat{q}_0(R)$ whenever R is Artinian. We call a map $q_0 : \mathbb{S}\mathrm{im}_0 \to \mathbb{B}$ a *quasi-slope*, if $\widehat{q}_0$ is everywhere defined. By abuse of terminology, we then also refer to this extension $q := \widehat{q}_0$ as a quasi-slope. In other words, $q : \mathbb{S}\mathrm{im} \to \mathbb{B}$ is a quasi-slope if, $q(J^nR)$ converges to $q(R)$, for every Noetherian local ring R. The following corollary is now immediate from Theorem 6.6.

Corollary 8.7. *Any continuous map* $q_0 : \mathbb{S}\mathrm{im}_0 \to \mathbb{B}$ *is a quasi-slope and its extension* $\widehat{q}_0$ *is a pre-slope.* □

We next show how some of the usual invariants, although in general not slopes, become quasi-slopes when properly modified. Let δ_0 be defined on $\mathbb{S}\mathrm{im}_0$ as follows. Given an Artinian local ring $(A, \mathfrak{m})$, let n be its degree of nilpotency (that is to say, the least k such that $\mathfrak{m}^k = 0$). Put

$$\delta_0(A) := \log_2\left(\frac{\ell(A)}{\ell(J^{n/2}A)}\right)$$

where for a positive real number r, we define $J^rA := J^zA$ with $z := \mathrm{int}(r)$ the largest integer less than or equal to r.

Proposition 8.8. *The map* δ_0 *is a quasi-slope. In fact,* $\widehat{\delta_0}(R)$ *is equal to the dimension of* R, *whenever this dimension is non-zero.*

Proof. Let R be a Noetherian local ring of dimension $d > 0$. By the Hilbert–Samuel theory, there exists a polynomial $P_R \in \mathbb{Q}[t]$ of degree d, such that $\ell(J^nR) = P_R(n)$ for $n \gg 0$. Hence $\delta_0(J^nR) = \log_2(P_R(n)/P_R(\mathrm{int}(n/2)))$, for $n \gg 0$. It is now an exercise to show that for any polynomial P of degree d, the limit of $P(n)/P(\mathrm{int}(n/2))$ is equal to 2^d. □

In view of this result, we call $\widehat{\delta}_0(R)$ the *quasi-dimension* of R. So only Artinian local rings have a quasi-dimension which is different from their (Krull) dimension. Using the formula

$$\lim_{n\to\infty} \frac{P(\mathrm{int}(\sqrt{n}))^2}{P(n)} = a_d$$

where a_d is the leading coefficient of a polynomial P, we get by a similar argument that the map ϵ_0 defined on $\mathbb{S}\mathrm{im}_0$ by the condition

$$\epsilon_0(A) := (\ell(J^{\sqrt{n}}A))^2/\ell(A)$$

is a quasi-slope, and $\widehat{\epsilon}_0(R) = e/d!$ whenever $d > 0$, where e is the multiplicity of R and d its dimension.

Several questions now arise naturally: what is the nature of the subset of $\mathbb{S}\mathrm{im}$ of all Noetherian local rings of a fixed quasi-slope? Can we break up (or even stratify) $\mathbb{S}\mathrm{im}$ in "natural" pieces on which a quasi-slope becomes continuous. Here is an example of how one can answer the second question for quasi-dimension. For a Noetherian local ring R, define $\rho(R)$ as the supremum over all n of

$$\left|\frac{d!\ell(J^nR)}{en^{d-1}} - n\right|$$

where $d := \dim(R)$ and $e := \mathrm{mult}(R)$ are respectively the dimension and multiplicity of R. In other words, $\rho(R)$ is the smallest real number $\rho \geq 0$ such that

$$n^d - \rho n^{d-1} \leq \frac{d!}{e}\ell(J^nR) \leq n^d + \rho n^{d-1} \tag{8}$$

for all $n > 0$. That this supremum exists is an easy consequence of the Hilbert–Samuel theory. For instance, if R is Artinian of length l, then $1 \leq \rho(R) \leq l$, but these bounds are not sharp. Note that ρ is *not* a quasi-slope (this is easily checked for $R := \kappa[[t]]$).

This new invariant determines the rate of convergence in the definition of the quasi-dimension, as the next result shows. To formulate it, we use $\ulcorner r \lrcorner$ to denote the rounding to the nearest integer of a real number r, that is to say, $\ulcorner r \lrcorner$ is the unique integer inside the half open interval $[r - \frac{1}{2}, r + \frac{1}{2})$.

Lemma 8.9. *For a Noetherian local ring R, if $n \geq 10\rho(R)$, then $\ulcorner \delta_0(J^nR) \lrcorner$ is equal to its dimension.*

Proof. Let $b := \rho(R)$, and let d and e be the respective dimension and multiplicity of R. Using (8), we get the estimates

$$1 - \frac{b}{n} \leq \frac{d!\ell(J^nR)}{en^d} \leq 1 + \frac{b}{n} \tag{9}$$

for all n. In the convergence of δ_0 we may take the limit over even n only, so let us assume that $n = 2m$. Dividing inequalities (9) for $2m$ by those for m, we get the estimates

$$2^d\left(\frac{1-\frac{b}{2m}}{1+\frac{b}{m}}\right) \leq \frac{\ell(J^{2m}R)}{\ell(J^mR)} = 2^{\delta_0(J^nR)} \leq 2^d\left(\frac{1+\frac{b}{2m}}{1-\frac{b}{m}}\right).$$

Hence, if the ratio between the two outside fractions is strictly less than 2, then after taking the logarithm with base two, they become the endpoints of an interval $[\alpha, \beta]$ of length strictly less than one, containing $\delta_0(J^nR)$. Since $\alpha < d < \beta$, the only integer in $[\alpha, \beta]$ is d, showing that $\ulcorner \delta_0(J^nR) \lrcorner = d$.

For the ratio to be at most 2, we need

$$\left(m + \frac{b}{2}\right)(m+b) < 2\left(m - \frac{b}{2}\right)(m-b)$$

and a simple calculation shows that this is true whenever $m > 5b$. □

Immediately from this we get:

Corollary 8.10. *For each $b \in \mathbb{N}$, let $\mathbb{S}\mathrm{im}_{\rho\leq b}$ be the subset of $\mathbb{S}\mathrm{im}$ consisting of all Noetherian local rings R such that $\rho(R) \leq b$, thus yielding a filtration $\mathbb{S}\mathrm{im}_{\rho\leq 0} \subseteq \mathbb{S}\mathrm{im}_{\rho\leq 1} \subseteq \ldots$ of $\mathbb{S}\mathrm{im}$. Then the quasi-dimension is a pre-slope on each $\mathbb{S}\mathrm{im}_{\rho\leq b}$.* □

A similar argument can be used to show that ϵ_0 is a pre-slope on each $\mathbb{S}\mathrm{im}_{\rho\leq b}$, by establishing an analogous bound for the convergence of ϵ_0 to $e/d!$ which only depends on $d := \dim(R)$, $e := \mathrm{mult}(R)$ and $\rho =: \rho(R)$. As far as bounding ρ itself is concerned, if R is Cohen–Macaulay, then it is bounded as a function of e and d only, but without the Cohen–Macaulay assumption this is probably false. In the latter case, we can use any "big degree" D à la Vasconcelos to arrive at such a bound in terms of d and $D(R)$ (this is an easy consequence of [17, Theorem 4.1]).

Recall (see, for instance, [7, III. Ex. 5.1]) that the *Euler characteristic* of a projective scheme X is defined by the formula

$$\chi(X) := \sum_i (-1)^i h^i(X, \mathcal{O}_X)$$

where $h^i(X, \mathcal{O}_X)$ is the dimension of the sheaf cohomology $H^i(X, \mathcal{O}_X)$ viewed as a vector space. In particular, if X is a curve, then $\chi(X) - 1$ is the *genus* of X.

Proposition 8.11. *The map $\mathbb{I}\mathrm{so}\mathbb{P}\mathrm{ol} \to \mathbb{R}$ sending a polarized scheme $\mathfrak{X} = (X, \mathcal{L})$ to $\chi(X)$ is a pre-slope.*

Proof. We calculate the Euler characteristic by means of the Hilbert–Samuel polynomial $P_X(n)$ of X as

$$\chi(X) = P_X(0). \tag{10}$$

In fact, as it is a birational invariant, we may calculate the Euler characteristic by means of any polarization $\mathfrak{X} = (X, \mathcal{L})$ of X. The *Hilbert series* of $\mathfrak{X}$ is defined as

$$\mathrm{Hilb}(\mathfrak{X}) := \sum_{n=0}^{\infty} h^0(X, \mathcal{L}^n)t^n.$$

Let $(R, \mathfrak{m}) := \mathrm{Vert}(\mathfrak{X})$. It is not hard to see that $H^0(X, \mathcal{L}^n) \cong \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and hence that X and R have the same Hilbert series and the same Hilbert–Samuel polynomial. Moreover, the connection between the Hilbert series $h(t)$ and the corresponding Hilbert–Samuel polynomial $P(n)$ is given by the formula

$$P(n) = \sum_{j=0}^{d-1} \frac{(-1)^j}{j!} \binom{n+d-1-j}{n} \frac{\partial^j}{\partial t}\left((1-t)^d h(t)\right)\Big|_{t=1}, \tag{11}$$

where d is the degree of P (that is to say, the dimension of X). Moreover, if h_i are Hilbert series with corresponding Hilbert polynomial P_i, then from the fact that

$$\frac{\partial^j}{\partial t}\left((1-t)^d t^n\right)\Big|_{t=1} = 0$$

for all $j < d$ and all n, we get from (11) that $P_1 = P_2$ whenever h_1 and h_2 are t-adically close. Therefore, by Proposition 8.2 and (10), the Euler characteristic is continuous. □

8.12. Motivic slopes

In work in progress [20,24], we assign to any Artinian local κ-algebra, with κ an algebraically closed field of size the continuum, its class in an abstract ring $\mathbf{G}$, called a *schemic* Grothendieck ring. This yields an injective map $\mathbb{I}\mathrm{so}\mathbb{A}\mathrm{rt}_0(\kappa) \to \mathbf{G}\colon R \mapsto [R]$, which is compatible with direct sum and tensor product. Hence, we may associate to any analytic germ R, its *motivic Hilbert series*

$$\mathrm{Hilb}^{\mathrm{form}}(R) := \sum_n [J^n R]t^n.$$

This yields a complete invariant with values in $\mathbf{G}[[t]]$. As $\mathbf{G}$ admits the classical Grothendieck ring of κ as a homomorphic image, it is necessarily a very complicated object. It is natural to ask if there exists however a countable residue ring $\bar{\mathbf{G}}$ of $\mathbf{G}$ such that distinct analytic germs still have distinct formal Hilbert series over $\bar{\mathbf{G}}$, thus yielding a slope (where we endow $\bar{\mathbf{G}}[[t]]$ with its t-adic topology).

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